NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 2)

2019 HSC Course Assessment Task 3 (Trial Examination)
27th of June, 2019

General instructions

- Working time – 3 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 13)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

○ 12M4A – Mr Ireland ○ 12M4B – Dr Jomaa ○ 12M4C – Miss Lee

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	**Total**
MARKS	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{100}$

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer sheet provided.

Questions **Marks**

1. Write $\dfrac{10i}{1+3i}$ in the form $a+ib$, where a and b are real. **1**

(A) $1-3i$ (B) $1+3i$ (C) $3+i$ (D) $3-i$

2. The equation $x^3+px+q=0$ has a double root 1. What is the value of $p+q$? **1**

(A) 1 (B) -1 (C) 2 (D) -3

3. The value of $i+i^2+i^3+...+i^{2019}$ **1**

(A) i (B) 1 (C) $-i$ (D) -1

4. The value of $\lim\limits_{h\to 0}\dfrac{f(a+3h)-f(a)}{h}$ is **1**

(A) $f^{'}(3a)$ (B) $3f^{'}(3a)$ (C) $3f^{'}(a)$ (D) $\dfrac{f^{'}(a)}{3}$

5. The eccentricity of the ellipse $\dfrac{x^2}{3^2+4^2}+\dfrac{y^2}{3^2}=1$ is **1**

(A) $\dfrac{3}{4}$ (B) $\dfrac{4}{9}$ (C) $\dfrac{4}{5}$ (D) $\dfrac{16}{25}$

6. $P(z)$ is a polynomial of degree 4. Which of the following statements must be false? **1**

(A) $P(z)$ has 4 real roots.

(B) $P(z)$ has 2 real and 2 non real roots

(C) $P(z)$ has 1 real and 3 non real roots

(D) $P(z)$ has no real roots

7. Given the curve $y = f(x)$, then the curve $y = f(|x|)$ is represented by **1**

(A) A reflection of $y = f(x)$ in the y axis.

(B) A reflection of $y = f(x)$ in the x axis.

(C) A reflection of $y = f(x)$ in the y axis for $x \geq 0$.

(D) A reflection of $y = f(x)$ in the x axis for $y \geq 0$.

8. How many vertical tangents can be drawn on the graph of $x^2 + y^2 + 4xy - 4 = 0$. **1**

(A) 1 (B) 2 (C) more than 2 (D) 0

9. Which of the following $\int \dfrac{dx}{\sqrt{7 - 6x - x^2}}$? **1**

(A) $\sin^{-1}(\dfrac{x-3}{2}) + c$

(B) $\sin^{-1}(\dfrac{x+3}{2}) + c$

(C) $\sin^{-1}(\dfrac{x-3}{4}) + c$

(D) $\sin^{-1}(\dfrac{x+3}{4}) + c$

10. Given $F(x) = \displaystyle\int_a^x (x - t)\cos 3t dt$, then $F''(x)$ is **1**

(A) $(1 - x)\cos 3x$

(B) $\sin 3x$

(C) $(1 - 3x)\cos 3x$

(D) $\cos 3x$

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) Find $\displaystyle\int x^4 e^{x^5+3}\,dx$ **1**

(b) Use the substitution $u = \dfrac{1}{1+x^2}$ to evaluate $\displaystyle\int \frac{dx}{x(1+x^2)^2}$. **2**

(c) Find $\displaystyle\int \frac{3x}{5x^2-4x+2}\,dx$ **3**

(d) Given $I = \displaystyle\int x\sin^2 x\,dx$ and $J = \displaystyle\int x\cos^2 x\,dx$

i. Show that $I + J = \dfrac{x^2}{2} + c_1$ **1**

ii. Find $J - I$ **3**

iii. Hence, or otherwise find I and J. **1**

(e)

i. Prove that if f is continuous function, then **2**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

ii. Hence, or otherwise show that **2**

$$\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx = \frac{\pi}{4}.$$

End of Question 11

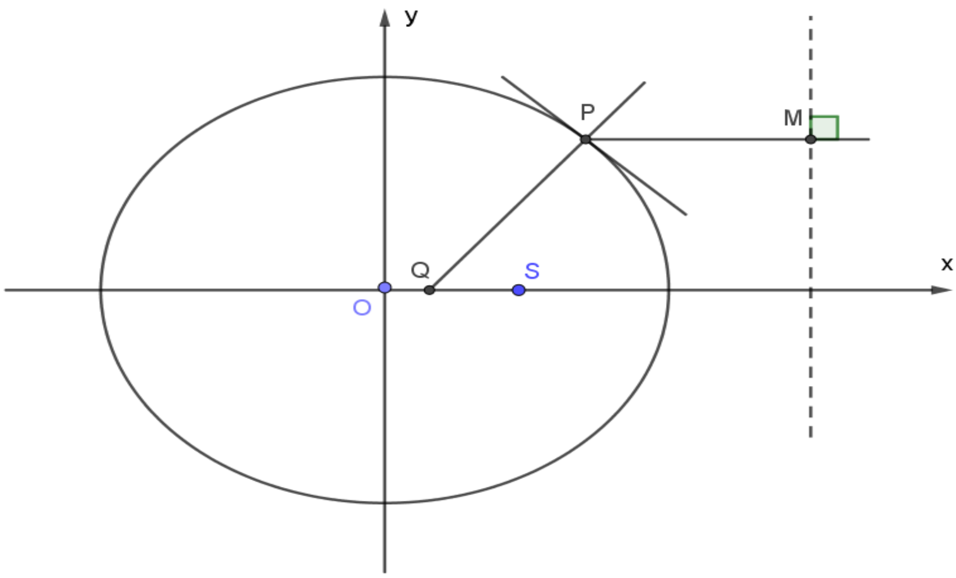
Question 12 (15 Marks) Commence a NEW page. **Marks**

(a) Given $z = 1 - i$, find the values of w such that $w^2 = i + 3\bar{z}$ **2**

(b) Find the Cartesian equation of the locus of a point P which represents the complex number z where $|z - 2i| = |z|$ **2**

(c) z is a point in the first quadrant of the Argand diagram which lies on the circle $|z - 3| = 3$. Given $\arg(z) = \theta$, find $\arg(z^2 - 9z + 18)$ in terms of θ. **2**

(d) Consider the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{7} = 1$



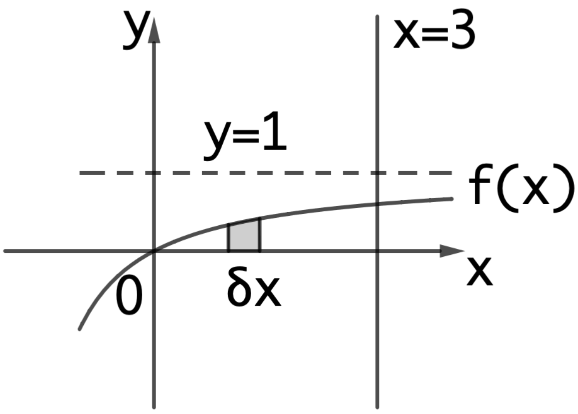
i. Write down the coordinates of the focus S and the equation of the associated directrix. **2**

ii. The equation of the normal to the ellipse at the point $P(x_1, y_1)$ is given by **2**

$$\frac{9x}{x_1} - \frac{7y}{y_1} = 2 \quad \text{(DO NOT PROVE THIS.)}$$

Let Q be the x-intercept of the normal and let M be the foot of the perpendicular from P to the directrix as shown in the diagram. Show that $QS = \dfrac{2}{9}PM$.

(e) The region bounded by the portion of the curve $f(x) = \dfrac{x}{x+2}$, and the x axis is rotated about the line $x = 3$.



i. Using the method of cylindrical shells, show that the volume of a typical shell at a distance x from the origin and with distance δx is given by **2**

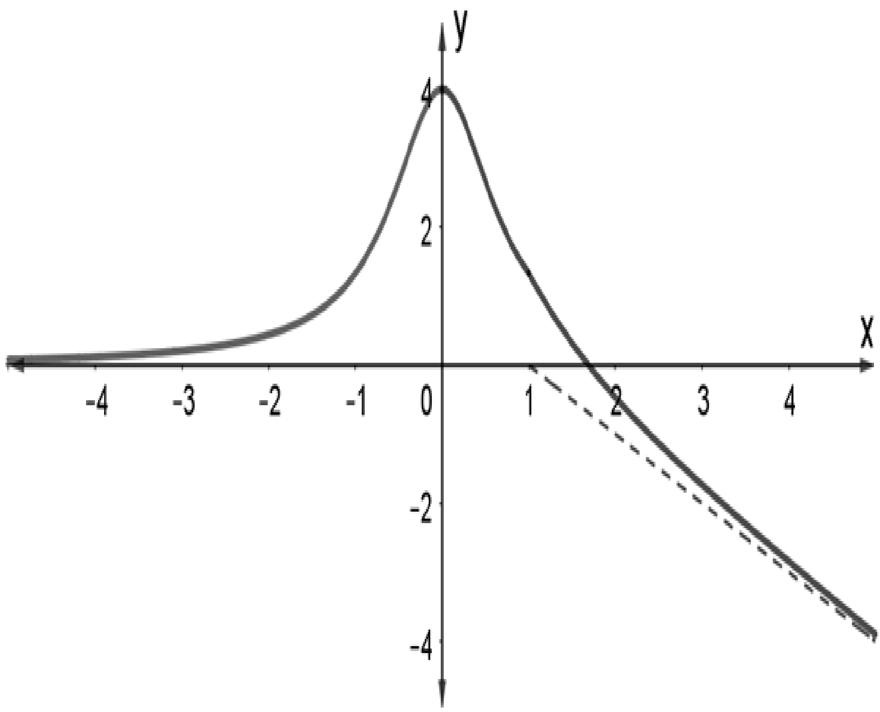
$$\delta V = 2\pi(3-x)\frac{x}{x+2}\delta x.$$

ii. Hence, find the volume of this solid. **3**

End of Question 12

Question 13 (15 Marks) Commence a NEW page. **Marks**

(a) The curve $y = f(x)$, sketched below, has asymptotes $y = 0$ and $y = 1 - x$.



Copy or trace the above graph onto three separate number planes. Use your diagrams to show sketches of the following graphs, showing all essential features clearly.

i. $y = [f(x)]^2$. **2**

ii. $|y| = f(x)$. **2**

iii. $y = \log(f(x))$. **2**

(b) α, β and γ are roots of the cubic equation $x^3 + qx + r = 0$.

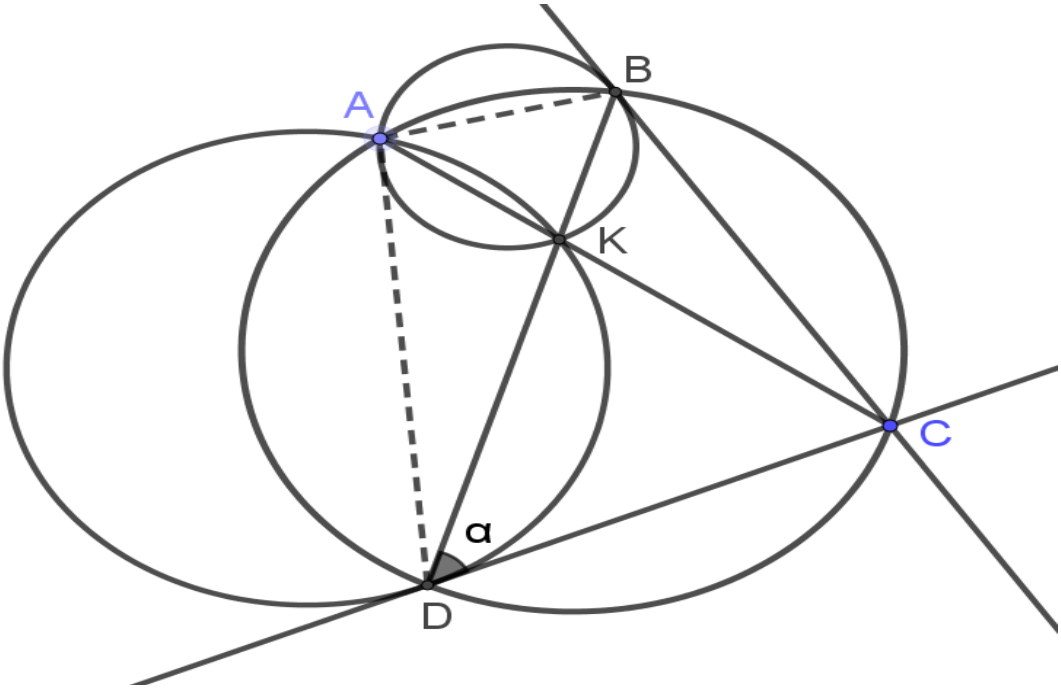
i. Find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ in terms of q and r. **2**

ii. Find the value of $\alpha^3 + \beta^3 + \gamma^3$ in terms of q and r **1**

iii. By considering $x = (\beta - \gamma)^2$, show that the equation whose roots are $(\beta - \gamma)^2$, $(\gamma - \alpha)^2$ and $(\alpha - \beta)^2$ is **2**

$$(x + q)^3 + 3q(x + q)^2 + 27r^2 = 0.$$

(c) In the diagram below, $ABCD$ is a cyclic quadrilateral and diagonals AC and BD intersect at K. Circles AKD and AKB are drawn and it is known that CD is a tangent to circle AKD. Let $\angle CDB = \alpha$.



i. Prove that BCD is isosceles. **2**

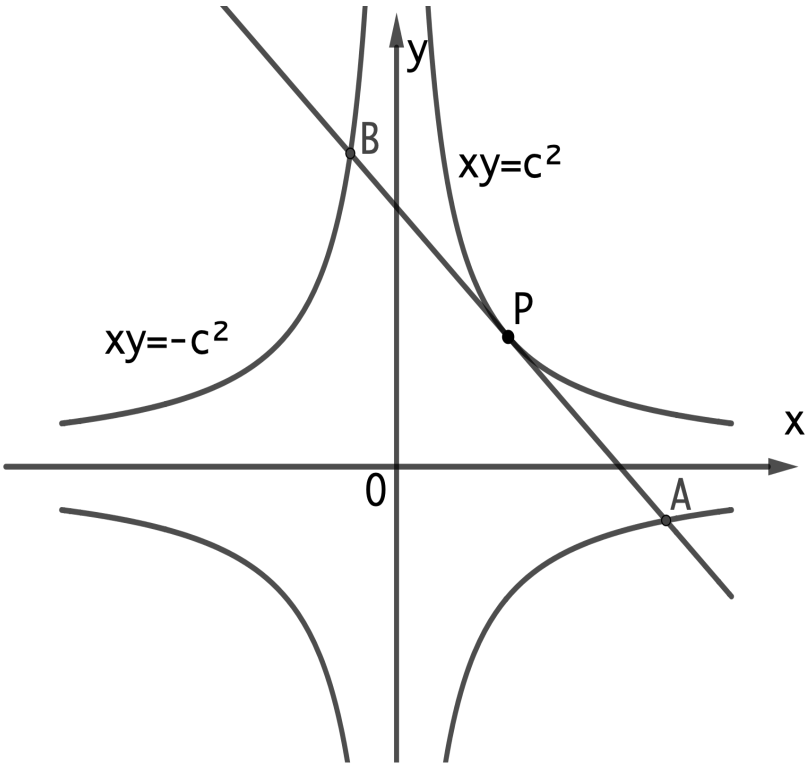
ii. Prove that CB is a tangent to circle AKB **2**

End of Question 13

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) Consider the rectangular hyperbolas $xy = c^2$ and $xy = -c^2$.
The point $P(cp, c/p)$ lies on $xy = c^2$ and the equation of tangent to $xy = c^2$ at point P is

$$x + p^2 y - 2cp = 0.$$



i. The tangent at P cuts the hyperbola $xy = -c^2$ at two points A and B. Show the coordinates of A and B are $\left(pc(1+\sqrt{2}), \dfrac{-c}{p(1+\sqrt{2})}\right)$ and $\left(pc(1-\sqrt{2}), \dfrac{-c}{p(1-\sqrt{2})}\right)$ respectively. **2**

ii. Show that the tangents to $xy = -c^2$ at A and B intersect at $Q(-cp, -c/p)$. **3**

iii. Hence, show that the area of the triangle ABQ is independent of p. **3**

(b) Consider the integral

$$I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}}\,dx.$$

i. Show that $I_n + I_{n-1} = \int_0^1 x^{n-1}\sqrt{x+1}\,dx$ **1**

ii. Use integration by parts to show that **2**

$$I_n = \frac{-2n}{2n+1} I_{n-1} + \frac{2\sqrt{2}}{2n+1}.$$

(c) Given that $x^3 + y^3 = 6xy$.

i. Find the tangent to $x^3 + y^3 = 6xy$ at the point $(3, 3)$. **2**

ii. At what point in the first quadrant is the tangent line horizontal? **2**

End of Question 14

Question 15 (15 Marks) Commence a NEW page. **Marks**

(a) Let α be a real number and suppose z is a complex number such that

$$z + \frac{1}{z} = 2\cos\alpha.$$

You may assume that

$$z^n + \frac{1}{z^n} = 2\cos n\alpha$$

for all positive integer n.

i. Let $\omega = z + \dfrac{1}{z}$. Prove that **2**

$$\omega^3 + \omega^2 - 2\omega - 2 = (z + \frac{1}{z}) + (z^2 + \frac{1}{z^2}) + (z^3 + \frac{1}{z^3})$$

ii. Hence, or otherwise, find all solutions of **3**

$$\cos\alpha + \cos 2\alpha + \cos 3\alpha = 0, \quad \text{for} \quad 0 \le \alpha \le 2\pi.$$

(b) A sequence x_n is defined by the following rules: $x_0 = 2a, x_1 = -a^2$ and **3**

$$x_{n+1} = -ax_n + a^2 x_{n-1} \quad \text{for} \quad n \ge 1.$$

Prove by mathematical induction that

$$x_n = a^{n+1}\left[\left(\frac{-1+\sqrt{5}}{2}\right)^n + \left(\frac{-1-\sqrt{5}}{2}\right)^n\right], \quad \text{for} \quad n \ge 0.$$

(c) Let a, b and c are positive real numbers.

Given that $x^2 + y^2 \ge 2xy$ for all positive real numbers x and y.

i. Prove that $a^2 + (bc)^2 \ge 2abc$ **1**

ii. Prove that $a^2 + b^2 + c^2 \ge ab + bc + ca$ **2**

iii. Prove that $a^2(1 + b^2) + b^2(1 + c^2) + c^2(1 + a^2) \ge 6abc$ **2**

iv. Hence, or otherwise prove that $a^2(1 + a^2) + b^2(1 + b^2) + c^2(1 + c^2) \ge 6abc$ **2**

End of Question 15

Question 16 (15 Marks) Commence a NEW page. **Marks**

The recurrence formula is defined by

$$T_0(x) = 2, \quad T_1(x) = 2\sec x, \quad T_2(x) = 4\sec^2 x - 2$$

and

$$T_k(x) = 2\sec x T_{k-1}(x) - T_{k-2}(x) \quad \text{for} \quad k \geq 2 \quad \text{and} \quad 0 \leq x < \frac{\pi}{2}.$$

i. Show that $T_3(x)$ and $T_4(x)$ are **2**

$$T_3(x) = 8\sec^3 x - 6\sec x$$

$$T_4(x) = 16\sec^4 x + 8\sec^2 x + 2$$

To find a formula for $T_k(x)$, let $F(Z)$ be the power series in Z with the coefficient of Z^k being $T_k(x)$. That is, let

$$F(Z) = 2 + 2\sec xZ + (4\sec^2 x - 2)Z^2 + \cdots + T_k(x)Z^k + \ldots$$

ii. Find $(1 - 2\sec xZ + Z^2)F(Z)$, hence show that **3**

$$F(Z) = \frac{2 - 2\sec xZ}{1 - 2\sec xZ + Z^2}$$

iii. Given that α and β are the zeros of $1 - 2\sec xZ + Z^2 = 0$. Show that **1**

$$1 - 2\sec xZ + Z^2 = \left(1 - \frac{Z}{\alpha}\right)\left(1 - \frac{Z}{\beta}\right)$$

iv. Using partial fraction, show that $F(Z)$ can be written in the form **2**

$$F(Z) = \frac{2 - 2\sec xZ}{1 - 2\sec xZ + Z^2} = \frac{A}{1 - \dfrac{Z}{\alpha}} + \frac{B}{1 - \dfrac{Z}{\beta}}$$

where A and B are constants.

v. For $|Z|$ sufficiently small, explain why $\dfrac{1}{1 - \dfrac{Z}{\alpha}}$ is equal to **2**

$$1 + \frac{Z}{\alpha} + \left(\frac{Z}{\alpha}\right)^2 + \cdots + \left(\frac{Z}{\alpha}\right)^k + \ldots,$$

hence show that the coefficient of $T_k(x)$ is $A\left(\dfrac{1}{\alpha}\right)^k + B\left(\dfrac{1}{\beta}\right)^k$

vi. Hence deduce that the formula for $T_k(x)$ is **3**

$$T_k(x) = \left(\frac{1}{\sec x + \tan x}\right)^k + \left(\frac{1}{\sec x - \tan x}\right)^k$$

vii. Find $\lim_{n\to\infty} \dfrac{T_{n+1}(x)}{T_n(x)}$ **2**

End of paper.

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

○ 12M4A – Mr Ireland ○ 12M4B – Dr Jomaa ○ 12M4C – Miss Lee

1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

5 – Ⓐ Ⓑ Ⓒ Ⓓ

6 – Ⓐ Ⓑ Ⓒ Ⓓ

7 – Ⓐ Ⓑ Ⓒ Ⓓ

8 – Ⓐ Ⓑ Ⓒ Ⓓ

9 – Ⓐ Ⓑ Ⓒ Ⓓ

10 – Ⓐ Ⓑ Ⓒ Ⓓ

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

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1 –	A	B	C	D
2 –	A	B	C	D
3 –	A	B	C	D
4 –	A	B	C	D
5 –	A	B	C	D
6 –	A	B	C	D
7 –	A	B	C	D
8 –	A	B	C	D
9 –	A	B	C	D
10 –	A	B	C	D

11.

(a) $\int x^4 e^{x^5+3} dx = \frac{1}{5}\int 5x^4 e^{x^5+3} dx = \frac{1}{5} e^{x^5+3} + c$

(b) $|z-2i| = |z|$

$|z-2i|^2 = |z|^2$

$x^2 + (y-2)^2 = x^2 + y^2$

$x^2 + y^2 - 4y + 4 = x^2 + y^2$

$4 - 4y = 0 \quad \therefore \boxed{y = 1}$

(c) $I = \int x \sin^2 x \, dx \quad , \quad J = \int x \cos^2 x \, dx$

(i) $I + J = \int x \sin^2 x \, dx + \int x \cos^2 x \, dx$

$= \int x(\sin^2 x + \cos^2 x) dx$

$= \int x \, dx$

$\boxed{I + J = \frac{x^2}{2} + C_1}$

(ii) $J - I = \int x \cos^2 x \, dx - \int x \sin^2 x \, dx$

$= \int x(\cos^2 x - \sin^2 x) \, dx$

$= \int x \cos 2x \, dx$

let $u = x$ and $\cos 2x \, dx = dv$

$du = dx$ and $v = \frac{1}{2}\sin 2x$

$\int x \cos 2x \, dx = \frac{1}{2} x \sin 2x - \frac{1}{2}\int \sin 2x \, dx$

$= \frac{1}{2} x \sin 2x - \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) + C_2$

$\boxed{J - I = \frac{1}{2} x \sin 2x + \frac{1}{4}\cos 2x + C_2}$

(iii) $$I + J = \frac{x^2}{2} + c_1 \qquad (1)$$

$$J - I = \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c_2 \qquad (2)$$

(1) + (2) $\therefore$ $$2J = \frac{x^2}{2} + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c_1 + c_2$$

$$\boxed{J = \frac{x^2}{4} + \frac{1}{4} x \sin 2x + \frac{1}{8} \cos 2x + C}$$

(1) − (2) $\therefore$ $$2I = \frac{x^2}{2} - \frac{1}{2} x \sin 2x - \frac{1}{4} \cos 2x + c_1 - c_2$$

$$\boxed{I = \frac{x^2}{4} - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + C}$$

(d) (i) $\int_0^a f(x)\,dx$

let $x = a-u$

$dx = -du$

$x=0, \quad u=a$

$x=a, \quad u=0$

$$\int_0^a f(x)\,dx = \int_a^0 f(a-u)(-du) = -\int_a^0 f(a-u)\,du$$

$$= \int_0^a f(a-u)\,du = \int_0^a f(a-x)\,dx.$$

(ii) $$I = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx = \int_0^{\pi/2} \frac{\cos^n(\pi/2 - x)}{\sin^n(\pi/2-x) + \cos^n(\pi/2-x)}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin^n(x)}{\cos^n x + \sin^n x}\,dx$$

$$2I = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx + \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx$$

$$= \int_0^{\pi/2} dx = \left[x\right]_0^{\pi/2} = \pi/2 - 0$$

$$I = \pi/4.$$

(e) $$\int \frac{3x}{5x^2-4x+2}\,dx = \frac{3}{10}\int \frac{10x-4+4}{5x^2-4x+2}\,dx$$

$$= \frac{3}{10}\int \frac{10x-4}{5x^2-4x+2}\,dx + \frac{6}{5}\int \frac{dx}{5x^2-4x+2}$$

$$= \frac{3}{10}\ln|5x^2-4x+2| + 6\int \frac{dx}{(5x)^2-20x+10}$$

$$= \frac{3}{10}\ln|5x^2-4x+2| + 6\int \frac{dx}{(5x-2)^2+6}$$

$$= \frac{3}{10}\ln|5x^2-4x+2| + \frac{6}{5}\int \frac{5\,dx}{(5x-2)^2+6}$$

$$= \frac{3}{10}\ln|5x^2-4x+2| + \frac{6}{5} \times \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{5x-2}{\sqrt{6}}\right) + C$$

$$= \frac{3}{10}\ln|5x^2-4x+2| + \frac{\sqrt{6}}{5}\tan^{-1}\left(\frac{5x-2}{\sqrt{6}}\right) + C$$

12.

(a) $z = 1 - i$

$w^2 = i + 3\bar{z}$

$= i + 3(1+i)$

$= 3 + 4i$

$= 2^2 - 1^2 + 2 \times 2 \times 1 \times i$

$= (2+i)^2$

$w = \pm(2+i)$

(b) $|z-3| = 3$

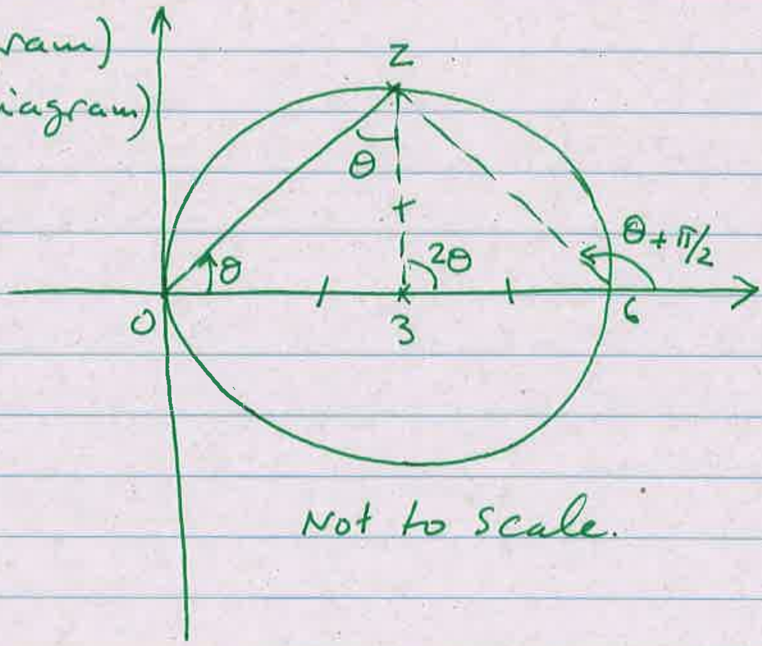
$\arg(z^2 - 9z + 18) = \arg((z-3)(z-6))$

$= \arg(z-3) + \arg(z-6)$

$\arg(z-3) = 2\theta$ (see diagram)

$\arg(z-6) = \theta + \pi/2$ (see diagram)

$\arg(z^2 - 9z + 18) = \pi/2 + 3\theta$



(c) $\dfrac{x^2}{9} + \dfrac{y^2}{7} = 1$

(i) $7 = 9(1-e^2) \therefore \dfrac{7}{9} = 1 - e^2 \therefore e^2 = 1 - \dfrac{7}{9} = \dfrac{2}{9} \therefore e = \dfrac{\sqrt{2}}{3}$

$a = 3$

$S(ae, 0) = (\sqrt{2}, 0)$

the directrix $x = \pm \dfrac{a}{e} = \pm \dfrac{3}{\frac{\sqrt{2}}{3}} = \pm \dfrac{9}{\sqrt{2}}$

(ii) $\frac{9x}{x_1} - \frac{7y}{y_1} = 2$

Sub $y = 0 \therefore x = \frac{2x_1}{9} \quad \therefore Q\left(\frac{2x_1}{9}, 0\right)$

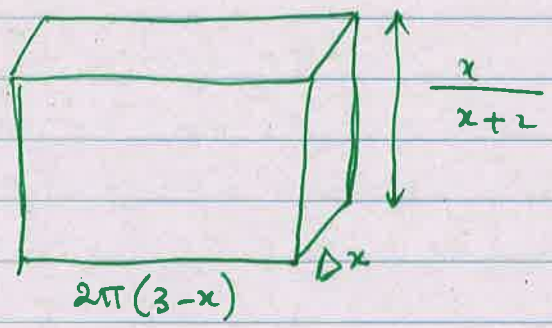
Also, $M\left(\frac{9}{\sqrt{2}}, y\right)$

$QS = \sqrt{2} - \frac{2x_1}{9}$

$PM = \frac{9}{\sqrt{2}} - x_1$

$QS = \sqrt{2} - \frac{2x_1}{9} = \frac{2}{9}\left(\frac{9}{\sqrt{2}} - x_1\right) = \frac{2}{9} PM$.

(d)



$$\text{Area} = 2\pi(3-x)\frac{x}{x+2}$$

$$\delta V = 2\pi(3-x)\frac{x}{x+2}\Delta x$$

$$V = \lim_{\Delta x \to 0} \delta V = \lim_{\Delta x \to 0} 2\pi(3-x)\frac{x}{x+2}\Delta x$$

$$V = 2\pi\int_0^3 (3-x)\frac{x}{x+2}\,dx = 2\pi\int_0^3 (3-x)\left(\frac{x+2-2}{x+2}\right)dx$$

$$= 2\pi\int_0^3 (3-x)\,dx + 4\pi\int_0^3 \frac{x-3}{x+2}\,dx$$

$$= 2\pi\int_0^3 (3-x)\,dx + 4\pi\int_0^3 dx - 20\pi\int_0 \frac{dx}{x+2}$$

$$= 2\pi\left[3x - \frac{x^2}{2}\right]_0^3 + 4\pi(x)_0^3 - 20\pi\left[\ln|x+2|\right]_0^3$$

continue (d)

$$V = 2\pi\left(9 - \frac{3^2}{2} - 0\right) + 4\pi(3-0) - 20\pi\left(\ln 5 - \ln 2\right)$$

$$= 9\pi + 12\pi - 20\pi \ln\left(\frac{5}{2}\right)$$

$$= 21\pi - 20\pi \ln\left(\frac{5}{2}\right)$$

$$\boxed{V = \pi\left(21 - 20\ln\frac{5}{2}\right)}$$

(e) $$\int \frac{dx}{x(1+x^2)}$$

let $u = \frac{1}{1+x^2}$ $\therefore$ $du = \frac{-2x}{(1+x^2)^2}dx$

$1+x^2 = \frac{1}{u}$ $\therefore$ $x^2 = \frac{1}{u} - 1 = \frac{1-u}{u}$ $\therefore$ $\frac{1}{x^2} = \frac{u}{1-u}$

$$\int \frac{dx}{x(1+x^2)} = -\frac{1}{2}\int \frac{-2x\,dx}{x^2(1+x^2)^2} = -\frac{1}{2}\int \frac{u\,du}{1-u}$$

$$= \frac{1}{2}\int \frac{1-u-1}{1-u}du = \frac{1}{2}\int du - \frac{1}{2}\int \frac{du}{1-u}$$

$$= \frac{1}{2}u + \frac{1}{2}\ln|1-u| + C$$

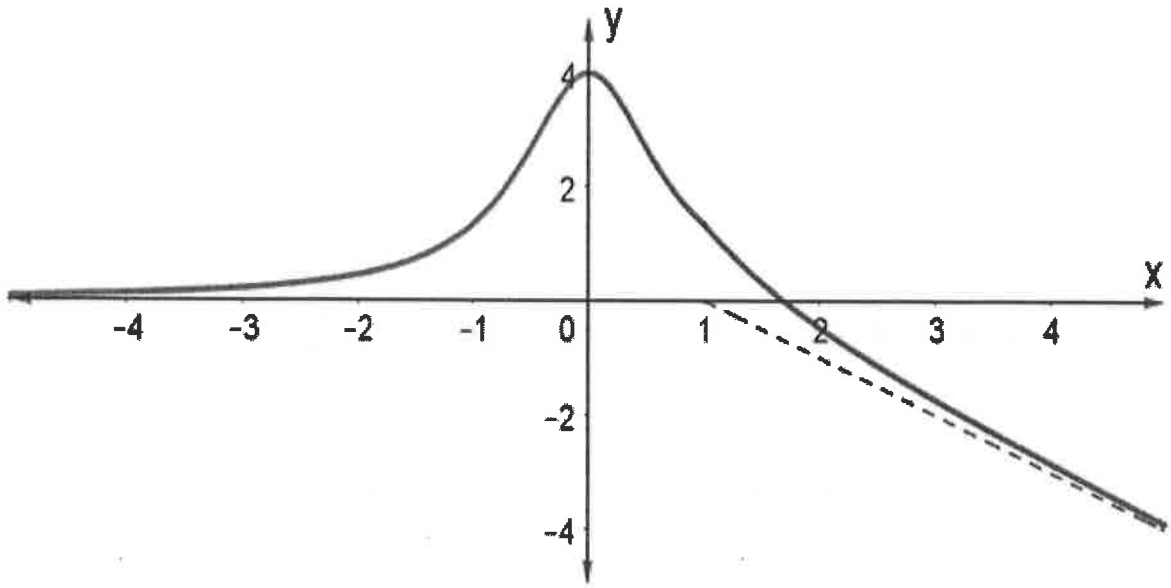
$$= \frac{1}{2} \times \frac{1}{1+x^2} + \frac{1}{2}\ln\left|1 - \frac{1}{1+x^2}\right| + C$$

$$= \frac{1}{2(1+x^2)} + \frac{1}{2}\ln\left(\frac{x^2}{1+x^2}\right) + C$$

Question 13 :

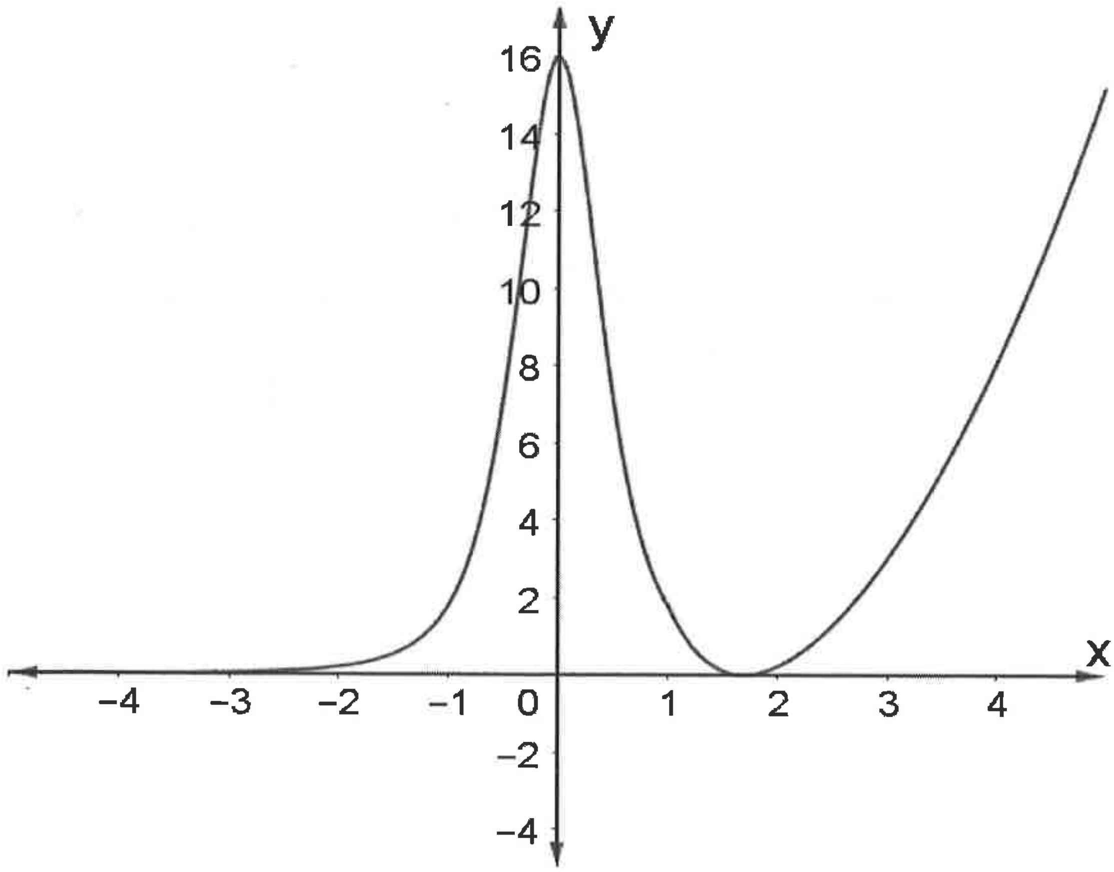
(a)

Y=f(x)

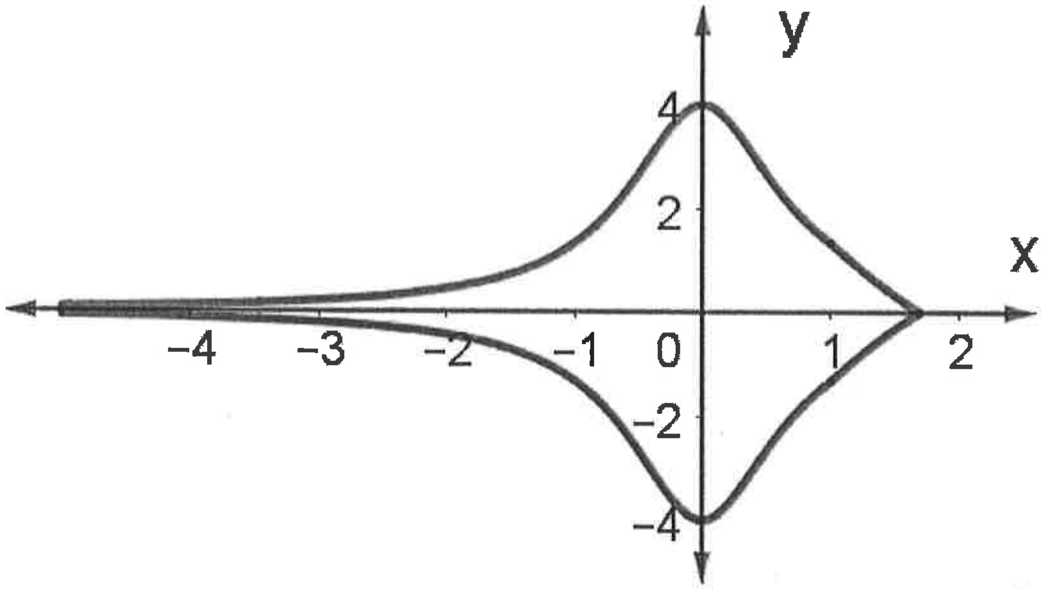


(i)

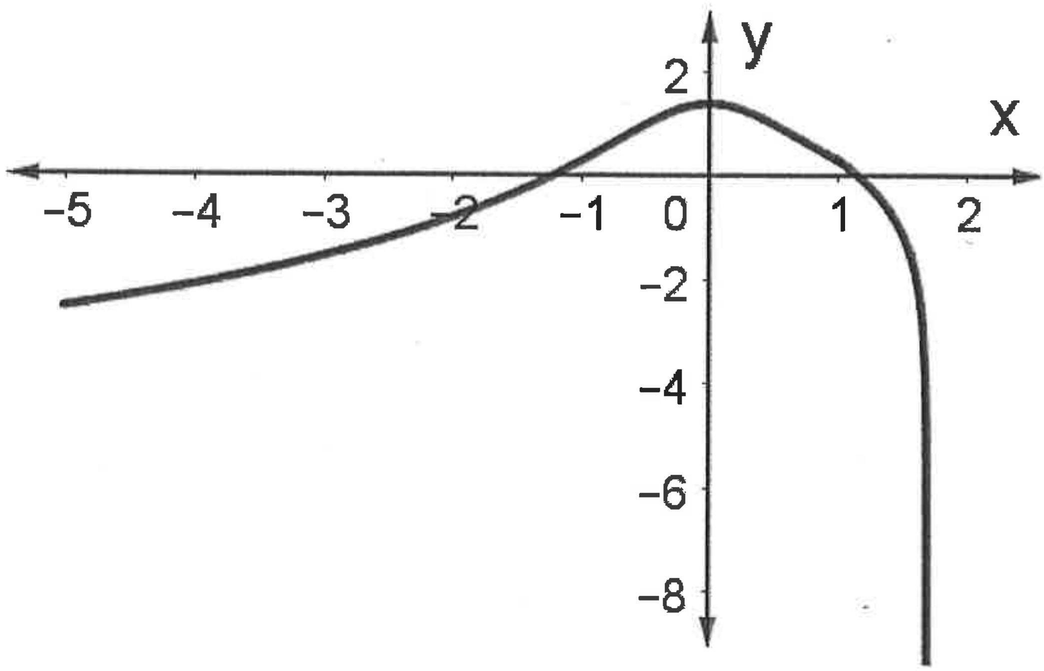
Y=f(x)^2



(ii) |y|=f(x)



(iii) Y=log(f(x))



(b) $x^3 + qx + r = 0$

α, β and γ are roots

$\therefore \quad \alpha + \beta + \gamma = 0 \quad (1)$

$\alpha\beta + \beta\gamma + \gamma\alpha = q \quad (2)$

$\alpha\beta\gamma = -r \quad (3)$

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{q}{-r} = -\frac{q}{r}$

(ii) $\alpha^3 + \beta^3 + \gamma^3 = -q(\alpha + \beta + \gamma) - 3r$

$= -q \times 0 - 3r$

$= -3r$

(iii) $x = (\beta - \gamma)^2 = (\beta + \gamma)^2 - 4\gamma\beta$

$(1) \therefore \beta + \gamma = -\alpha$

$(3) \therefore \beta\gamma = -r/\alpha$

$x = (-\alpha)^2 - 4(-r/\alpha) = \alpha^2 + \frac{4r}{\alpha} \quad \therefore \alpha^3 - \alpha x + 4r = 0 \quad (4)$

but, $\alpha^3 + q\alpha + r = 0 \quad (5)$

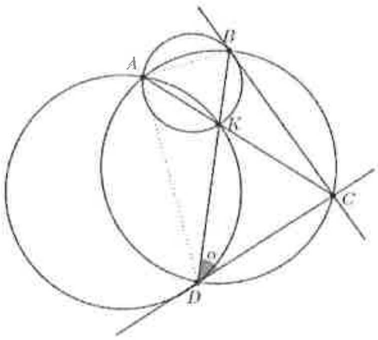
$(5) - (4) \quad \therefore \quad (q + x)\alpha = 3r \quad \therefore \quad \alpha = \frac{3r}{q + x} \quad (6)$

sub (6) into (5) we obtain:

$$\frac{27r^3}{(q+x)^3} + \frac{3qr}{q+x} + r = 0$$

$$27r^3 + 3q(q+x)^2 + (q+x)^3 = 0$$

OR $\quad x^3 + 6qx^2 + 9q^2x + 4q^3 + 27r^2 = 0.$



In the diagram above, $ABCD$ is a cyclic quadrilateral and diagonals AC and BD intersect at K. Circles AKD and AKB are drawn and it is known that CD is a tangent to circle AKD. Let $\angle CDB = \alpha$.

Use the separate blue answer sheet for Question 16 (b).

(i) Prove that ΔBCD is isosceles. **2**

(ii) Prove that CB is a tangent to circle AKB **2**

(c) (i) If $\angle CDB = \alpha$

then, $\angle DAC = \alpha$ (Angle in the alternate segment theorem)

and $\therefore$ $\angle DBC = \alpha$ (Angles on the circumference standing on the same arc are equal)

$\therefore$ ΔBCD is isosceles (Base $\angle$'s equal).

(ii) $CD^2 = AC \cdot CK$ (Square on tangent).

but $CD = BC$ (Equal sides of isosceles ΔBCD)

$\therefore$ $BC^2 = AC \cdot CK$

$\therefore$ BC must be a tangent to circle AKB.

(14)

(a) $x^3 + y^3 = 6xy$

(i) $3x^2 + 3y^2 \dfrac{dy}{dx} = 6y + 6x \dfrac{dy}{dx}$

$x^2 + y^2 \dfrac{dy}{dx} = 2y + 2x \dfrac{dy}{dx}$

$(y^2 - 2x)\dfrac{dy}{dx} = 2y - x^2$

$\dfrac{dy}{dx} = \dfrac{2y - x^2}{y^2 - 2x}$

at $(3,3)$, $\dfrac{dy}{dx} = \dfrac{2(3) - 3^2}{3^2 - 2(3)} = \dfrac{-3}{3} = -1$

The equation of tangent is

$y - 3 = -1(x - 3) \quad \therefore \quad \boxed{y = -x + 6}$

(ii) Tangent line is horizontal $\therefore$ $y' = 0$

$\therefore \; 2y - x^2 = 0 \quad \therefore \quad y = x^2/2$

$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\left(\frac{x^2}{2}\right)$

$x^3 + \dfrac{x^6}{8} = 3x^3$

$x^6 = 16x^3$

$x^3 = 16$

$x = 16^{1/3} = 2^{4/3}$

$y = \dfrac{(2^{4/3})^2}{2} = 2^{5/3}$

$\therefore$ At $(2^{4/3}, 2^{5/3})$ the tangent line is horizontal.

(b) (i) $x + p^2 y - 2cp = 0$

Multiply by x, we obtain

$$x^2 + p^2 xy - 2cpx = 0, \text{ but } xy = -c^2$$

$$x^2 - c^2p^2 - 2cpx = 0$$

$$x^2 - 2cpx + c^2p^2 - 2c^2p^2 = 0$$

$$(x - cp)^2 = 2c^2p^2$$

$$x - cp = \pm cp\sqrt{2}$$

$$x = cp(1 \pm \sqrt{2})$$

$\therefore$ $A\left(cp(1+\sqrt{2}), \dfrac{-c}{p(1+\sqrt{2})}\right)$

and $B\left(cp(1-\sqrt{2}), \dfrac{-c}{p(1-\sqrt{2})}\right)$

(ii) The equation of tangent at A is

$$y + \frac{c}{p(1+\sqrt{2})} = \frac{c^2}{c^2p^2(1+\sqrt{2})^2}\left(x - cp(1+\sqrt{2})\right)$$

$$\boxed{y = \frac{1}{p^2(3+2\sqrt{2})}x - \frac{2c}{p(1+\sqrt{2})}} \quad (1)$$

Similarly at B is

$$\boxed{y = \frac{1}{p^2(3-2\sqrt{2})}x - \frac{2c}{p(1-\sqrt{2})}} \quad (2)$$

$$(1)-(2) \therefore 0 = \frac{x}{p^2}\left(\frac{1}{3+2\sqrt{2}} - \frac{1}{3-2\sqrt{2}}\right) - \frac{2c}{p}\left(\frac{1}{1+\sqrt{2}} - \frac{1}{1-\sqrt{2}}\right)$$

$$= \frac{x}{p^2}\left(\frac{-4\sqrt{2}}{1}\right) - \frac{2c}{p}\left(\frac{-2\sqrt{2}}{-1}\right)$$

$$= \frac{-4\sqrt{2}}{p^2}x - \frac{4\sqrt{2}c}{p} \quad \therefore \boxed{x = -cp}$$

(b) continue

use (1)

$$y = \frac{1}{p^2(3+2\sqrt{2})}(-cp) - \frac{2c}{p(1+\sqrt{2})}$$

$$= \frac{-c}{p} \times \frac{3-2\sqrt{2}}{9-8} - \frac{2c(1-\sqrt{2})}{p(1-2)}$$

$$= -\frac{c}{p}\left(3-2\sqrt{2}-2+2\sqrt{2}\right) = -\frac{c}{p}$$

$$\therefore \quad Q\left(-cp, -c/p\right) \quad \text{as required.}$$

(iii) $$AB = \sqrt{\left[cp(1+\sqrt{2}) - cp(1-\sqrt{2})\right]^2 + \left[\frac{-c}{p(1+\sqrt{2})} - \frac{-c}{p(1-\sqrt{2})}\right]^2}$$

$$= \sqrt{8p^2c^2 + \frac{8c^2}{p^2}}$$

$$= 2\sqrt{2}\,c\sqrt{p^2 + \frac{1}{p^2}}$$

Equation of AB is $x^2 + p^2y - 2cp = 0$

perpendicular distance from Q to AB is

$$\frac{\left|-cp + p^2(-c/p) - 2cp\right|}{\sqrt{1+p^4}} = \frac{4cp}{\sqrt{p^4+1}}$$

$$\text{Area of } ABQ = \frac{1}{2} \times \frac{4cp}{\sqrt{p^4+1}} \times 2\sqrt{2}\,c\sqrt{p^2+\frac{1}{p^2}}$$

$$= 4c^2\sqrt{2}$$

is independent of p as required.

6 (i) $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}}\,dx$

$I_{n-1} = \int_0^1 \frac{x^{n-1}}{\sqrt{x+1}}\,dx$

$I_n + I_{n-1} = \int_0^1 \frac{x^{n-1}(1+x)}{\sqrt{x+1}}\,dx$

$= \int_0^1 x^{n-1}\sqrt{1+x}\,dx.$

(ii) let $x^{n-1}\,dx = dv \quad \therefore \quad v = \frac{1}{n}x^n$

$u = \sqrt{1+x} \quad \therefore \quad du = \frac{1}{2\sqrt{1+x}}\,dx$

$\int_0^1 x^{n-1}\sqrt{1+x}\,dx = \left[\frac{1}{n}x^n\sqrt{1+x}\right]_0^1 - \frac{1}{2n}\int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$

$= \frac{1}{n}\sqrt{2} - \frac{1}{2n} I_n$

$I_n + I_{n-1} = \frac{1}{n}\sqrt{2} - \frac{1}{2n} I_n$

$2n I_n + 2n I_{n-1} = 2\sqrt{2} - I_n$

$(2n+1) I_n = -2n I_{n-1} + 2\sqrt{2}$

$I_n = \frac{-2n}{2n+1} I_{n-1} + \frac{2\sqrt{2}}{2n+1}.$

(15)

(i) $w^3 + w^2 - 2w - 2$

$= w^2(w+1) - 2(w+1)$

$= (w+1)(w^2-2)$

$= \left(z+\frac{1}{z}+1\right)\left(\left(z+\frac{1}{z}\right)^2-2\right)$

$= \left(z+\frac{1}{z}+1\right)\left(z^2+\frac{1}{z^2}\right)$

$= z^3 + \frac{1}{z} + z + \frac{1}{z^3} + z^2 + \frac{1}{z^2}$

$= z + \frac{1}{z} + z^2 + \frac{1}{z^2} + z^3 + \frac{1}{z^3}$.

(ii) $z + \frac{1}{z} = 2\cos\alpha$

$z^2 + \frac{1}{z^2} = 2\cos 2\alpha$

$z^3 + \frac{1}{z^3} = 2\cos 3\alpha$

$w^3 + w^2 - 2w - 2 = (w+1)(w^2-2)$

$= (2\cos\alpha + 1)(2\cos 2\alpha)$

$= 2\cos 2\alpha(2\cos\alpha + 1)$

$\cos\alpha + \cos 2\alpha + \cos 3\alpha = 0 \quad \therefore \frac{1}{2}\left[z + \frac{1}{z} + z^2 + \frac{1}{z^2} + z^3 + \frac{1}{z^3}\right] = 0$

$\therefore \frac{1}{2}(w+1)(w^2-2) = 0$

$\therefore \cos 2\alpha(2\cos\alpha + 1) = 0$

$\cos 2\alpha = 0$ or $2\cos\alpha + 1 = 0$

$2\alpha = \pi/2$ or $\frac{3\pi}{2}$ or $\frac{5\pi}{2}$ or $\frac{7\pi}{2}$ or $\cos\alpha = -\frac{1}{2}$ $\therefore \alpha = 2\pi/3$ or $4\pi/3$

$$\boxed{\alpha = \pi/4, \frac{3\pi}{4}, \frac{5\pi}{4}, 7\pi/4, \frac{2\pi}{3}, \frac{4\pi}{3}.}$$

(b) $x_0 = a^{0+1}\left[\left(\frac{-1+\sqrt5}{2}\right)^0+\left(\frac{-1-\sqrt5}{2}\right)^0\right] = a(1+1) = 2a$

$x_1 = a^2\left[\frac{-1+\sqrt5}{2}+\frac{-1-\sqrt5}{2}\right] = a^2\left(-\frac12-\frac12\right) = -a^2.$

Assume it is true for $n=k-1$ and $n=k$

$$\left.\begin{aligned} x_{k-1} &= a^k\left[\left(\frac{-1+\sqrt5}{2}\right)^{k-1}+\left(\frac{-1-\sqrt5}{2}\right)^{k-1}\right] \\ x_k &= a^{k+1}\left[\left(\frac{-1+\sqrt5}{2}\right)^{k}+\left(\frac{-1-\sqrt5}{2}\right)^{k}\right]\end{aligned}\right\} (*) \text{ (induction hypothesis)}$$

Prove it true for $n=k+1$.

$$\begin{aligned} x_{k+1} &= -a x_k + a^2 x_{k-1} \\ &= -a\,a^{k+1}\left[\left(\frac{-1+\sqrt5}{2}\right)^{k}+\left(\frac{-1-\sqrt5}{2}\right)^{k}\right] + a^2a^k\left[\left(\frac{-1+\sqrt5}{2}\right)^{k-1}+\left(\frac{-1-\sqrt5}{2}\right)^{k-1}\right] \\ &= a^{k+2}\left[\left(\frac{-1+\sqrt5}{2}\right)^{k-1}\left(1-\frac{-1+\sqrt5}{2}\right)+\left(\frac{-1-\sqrt5}{2}\right)^{k-1}\left(1-\frac{-1-\sqrt5}{2}\right)\right] \\ &= a^{k+2}\left[\left(\frac{-1+\sqrt5}{2}\right)^{k-1}\left(\frac{3-\sqrt5}{2}\right)+\left(\frac{-1-\sqrt5}{2}\right)^{k-1}\left(\frac{3+\sqrt5}{2}\right)\right]\end{aligned}$$

$$\boxed{\begin{aligned}\left(\frac{-1+\sqrt5}{2}\right)^2 &= \frac{1+5-2\sqrt5}{4} = \frac{6-2\sqrt5}{4} = \frac{3-\sqrt5}{2} \\ \left(\frac{-1-\sqrt5}{2}\right)^2 &= \frac{1+5+2\sqrt5}{4} = \frac{6+2\sqrt5}{4} = \frac{3+\sqrt5}{2}.\end{aligned}}$$

$$\begin{aligned} x_{n+1} &= a^{k+2}\left[\left(\frac{-1+\sqrt5}{2}\right)^{k-1}\left(\frac{-1+\sqrt5}{2}\right)^2+\left(\frac{-1-\sqrt5}{2}\right)^{k-1}\left(\frac{-1-\sqrt5}{2}\right)^2\right] \\ &= a^{k+2}\left[\left(\frac{-1+\sqrt5}{2}\right)^{k+1}+\left(\frac{-1-\sqrt5}{2}\right)^{k+1}\right]\end{aligned}$$

Hence by mathematical induction, its is true for all $n \geq 0$.

(c) $x^2+y^2 \geq 2xy$

(i) $a^2+(bc)^2 \geq 2a(bc) = 2abc.$

(ii) $\left.\begin{aligned} a^2+b^2 &\geq 2ab \\ a^2+c^2 &\geq 2ac \\ b^2+c^2 &\geq 2bc \end{aligned}\right\}$ $a^2+b^2+c^2 \geq ab+bc+ca.$

(iii) $a^2(1+b^2)+b^2(1+c^2)+c^2(1+a^2)$

$= a^2+a^2b^2+b^2+b^2c^2+c^2+c^2a^2$

$= a^2+b^2+c^2+a^2b^2+b^2c^2+c^2a^2$

$\geq 6abc.$

$a^2+b^2c^2 \geq 2abc$
$b^2+a^2c^2 \geq 2abc$
$c^2+a^2b^2 \geq 2abc$

(iv) $a^4+b^4+c^4+a^2+b^2+c^2$

$\geq a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2$

$\geq 6abc.$

Question 16:

$T_0(x) = 2$, $T_1(x) = 2\sec x$, $T_2(x) = 4\sec^2 x - 2$

$T_k(x) = 2\sec x\, T_{k-1}(x) - T_{k-2}(x) \quad k \geq 2 \;, \quad 0 \leq x < \pi/2 \quad -(*)$

(i) $T_3(x) = 2\sec x\, T_2(x) - T_1(x)$

$= 2\sec x\,(4\sec^2 x - 2) - 2\sec x$

$= 8\sec^3 x - 4\sec x - 2\sec x$

$= 8\sec^3 x - 6\sec x$

$T_4(x) = 2\sec x\, T_3(x) - T_2(x)$

$= 2\sec x\,(8\sec^3 x - 6\sec x) - 4\sec^2 x + 2$

$= 16\sec^4 x - 12\sec^2 x - 4\sec^2 x + 2$

$= 16\sec^4 x - 16\sec^2 x + 2$

(ii) $F(z) = 2 + 2\sec x\, z + (4\sec^2 x - 2)z^2 + \dots + T_k(x) z^k + \dots$

$-2\sec x\, z\, F(z) = \quad -4\sec x\, z - 4\sec^2 x\, z^2 - 2\sec x\, z^3 T_2(x) + \dots - 2\sec x\, T_{k-1}(x) z^k + \dots$

$z^2 F(z) = \quad 2z^2 + T_1(x) z^3 + \dots + T_{k-2}(x) z^k + \dots$

$(1 - 2\sec x\, z + z^2)F(z) = 2 - 2\sec x\, z + 0z^2 + (T_3(x) - 2\sec x\, T_2(x) + T_1(x))z^3 + \dots$

$+ \left(T_k(x) - 2\sec x\, T_{k-1}(x) + T_{k-2}(x)\right) z^k + \dots$

$= 2 - 2\sec x\, z$

Coefficients of z^k, $k \geq 3$ are zero because they satisfy the recurrence formula $(*)$

$$\therefore \quad F(z) = \frac{2 - 2\sec x\, z}{1 - 2\sec x\, z + z^2}$$

(iii) α, β are the roots of $1 - 2\sec x\, z + z^2 = 0 \quad \therefore \alpha\beta = 1$

$1 - 2\sec x\, z + z^2 = (z - \alpha)(z - \beta) = \dfrac{(z-\alpha)(z-\beta)}{\alpha\beta}$

$= \left(\dfrac{z}{\alpha} - 1\right)\left(\dfrac{z}{\beta} - 1\right)$

$= \left(1 - \dfrac{z}{\alpha}\right)\left(1 - \dfrac{z}{\beta}\right).$

using partial fraction.

(ii) $F(z) = \dfrac{2-2\sec x\, z}{1-2\sec x\, z+z^2} = \dfrac{2-2\sec x\, z}{(1-\frac{z}{\alpha})(1-\frac{z}{\beta})} = \dfrac{A}{1-\frac{z}{\alpha}} + \dfrac{B}{1-\frac{z}{\beta}}$

$= \dfrac{A(1-\frac{z}{\beta}) + B(1-\frac{z}{\alpha})}{(1-\frac{z}{\alpha})(1-\frac{z}{\beta})} = \dfrac{A+B-(\frac{A}{\beta}+\frac{B}{\alpha})z}{(1-\frac{z}{\alpha})(1-\frac{z}{\beta})}$

$\therefore \quad A+B = 2 \quad \therefore B = 2-A \quad (1)$

$\dfrac{A}{\beta} + \dfrac{B}{\alpha} = 2\sec x \quad \therefore \quad \alpha A + \beta B = 2\sec x \ (2) \quad (\alpha\beta = 1)$

$\alpha + \beta = 2\sec x$ (sum of roots)

$\therefore \quad A = B = 1.$

(v). $|z| \ll$, ~~$\dfrac{1}{1-\frac{z}{\alpha}}$ is the limiting sum~~ of the infinite geometric series

$1 + \dfrac{z}{\alpha} + \left(\dfrac{z}{\alpha}\right)^2 + \cdots + \left(\dfrac{z}{\alpha}\right)^k + \cdots$

$F(z) = \dfrac{A}{1-\frac{z}{\alpha}} + \dfrac{B}{1-\frac{z}{\beta}} = A\left(1 + \dfrac{z}{\alpha} + \left(\dfrac{z}{\alpha}\right)^2 + \cdots + \left(\dfrac{z}{\alpha}\right)^k + \cdots + \right)$

$B\left(1 + \dfrac{z}{\beta} + \left(\dfrac{z}{\beta}\right)^2 + \cdots + \left(\dfrac{z}{\beta}\right)^k + \cdots\right)$

But $A = B = 1$

$= 2 + \left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)z + \left(\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}\right)z^2 + \cdots + \left(\left(\dfrac{1}{\alpha}\right)^k + \left(\dfrac{1}{\beta}\right)^k\right)z^k + \cdots$

$\therefore \quad T_k(x) = \left(\dfrac{1}{\alpha}\right)^k + \left(\dfrac{1}{\beta}\right)^k$

(vi) From the quadratic equation $1-2\sec x\, z+z^2=0$

$\alpha = \dfrac{2\sec x + \sqrt{4\sec^2 x - 4}}{2} = \sec x + \tan x$

$\beta = \dfrac{2\sec x - \sqrt{4\sec^2 x - 4}}{2} = \sec x - \tan x$

$$T_k(x) = \left(\frac{1}{\sec x+\tan x}\right)^k + \left(\frac{1}{\sec x-\tan x}\right)^k$$

$$= (\sec x - \tan x)^k + (\sec x + \tan x)^k$$

(vii) $\lim_{n\to\infty} \frac{T_{n+1}(x)}{T_n(x)}$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{\sec x+\tan x}\right)^{n+1} + \left(\frac{1}{\sec x-\tan x}\right)^{n+1}}{\left(\frac{1}{\sec x+\tan x}\right)^{n} + \left(\frac{1}{\sec x-\tan x}\right)^{n}}$$

$$= \lim_{n\to\infty} \frac{(\sec x+\tan x)^{n+1} + (\sec x-\tan x)^{n+1}}{(\sec x+\tan x)^{n} + (\sec x-\tan x)^{n}}$$

$$= \lim_{n\to\infty} \frac{(\sec x+\tan x)^{n+1}\left[1+\left(\frac{\sec x-\tan x}{\sec x+\tan x}\right)^{n+1}\right]}{(\sec x+\tan x)^{n}\left[1+\left(\frac{\sec x-\tan x}{\sec x+\tan x}\right)^{n}\right]}$$

$$\boxed{\frac{\sec x-\tan x}{\sec x+\tan x} < 1, \text{ As } n\to\infty, \left(\frac{\sec x-\tan x}{\sec x+\tan x}\right)^{n} \to 0}$$

$$\therefore \lim_{n\to\infty} \frac{T_{n+1}(x)}{T_n(x)} = \sec x + \tan x.$$